THE LONGITUDINAL DEVELOPMENT OF UNIT CONCEPTS IN AREA AND VOLUME MEASUREMENT CONTEXTS: A CASE STUDY

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This report describes the longitudinal development of one student, Drew, in terms of his unit concepts in area and volume measurement situations across Grades 2 through 5. Data were collected during individual interviews and an open-response assessment within the context of a four-year longitudinal teaching experiment. Results indicate that Drew's use of unit developed over time with respect to identifying and operating on composite units. In addition, his understanding of the conceptual link between the iteration of a composite unit and the application of multiplication in both area and volume measurement contexts developed gradually. Furthermore, changes in his use of unit in an area measurement context often precipitated changes in his use of unit in a volume measurement context.

Keywords: Measurement, Geometry and Geometrical and Spatial Thinking, Learning Trajectories

The development of children's conceptions of spatial measurement has attracted the attention of researchers. However, much of what is known has been studied using cross sectional studies and has focused on only one domain of measurement in isolation (i.e., Battista & Clements, 1996; Battista, Clements, Arnoff, Battista, & Borrow, 1998; Outhred & Mitchelmore, 2000). Therefore, little is known about how children's sense of unit develops over time and connects across domains of spatial measurement. The purpose of this report is to address this gap in the literature by describing the longitudinal growth of one student, Drew, with respect to his developing sense of unit in the contexts of both area and volume across Grades 2 through 5.

Drew gained sophistication in his thinking with respect to unit across area and volume measurement contexts; however, his longitudinal growth was gradual. By closely studying Drew's evolving sense of unit for both area and volume, we gained insights into how concepts related to unit develop over time and across domains of measurement, especially for students who may struggle to coordinate area and volume.

Theoretical Framework

Our research team conducted a longitudinal study of the development of children's spatial measurement concepts across Grades 2 through 5 from a hierarchic interactionalist perspective (Clements & Sarama, 2007), which is a theoretical framework that synthesizes empiricism, (neo)nativism, and interactionalism. At the core of Sarama and Clements' (2009) theoretical and empirical frameworks are hypothetical learning trajectories (LTs). An LT consists of three components: 1) a specific learning goal in a mathematical content domain, 2) a likely path for

learning, and 3) the instructional tasks specifically designed to engender the mental processes or actions that move students along that path (Sarama & Clements, 2009).

Purpose

The purpose of the longitudinal study was to refine and revise LTs for length, area, and volume measurement. Specifically, we posed the question: How do students develop coherent knowledge and integrated strategies for measurement across the developmental span from Grade 2 through 5? The goal of this paper is to describe how children's concepts of units develop across contexts of area and volume from Grade 2 through 5 by focusing on Drew as a case study.

Methodology

This report focuses on one participant (who is a member of a larger sample of 16 children) at a public school in the Midwest. The data was collected between March of 2008 and May of 2011 within the context of a four-year, longitudinal teaching experiment (Steffe & Thompson, 2000). We used previously developed LTs (Sarama & Clements, 2009) to inform our task development and data analysis as we aimed to investigate children's conceptions of spatial measurement across Grades 2 to 5. Results are drawn from semi-structured interviews of approximately 20 to 30 minutes in length and one open-response assessment. Each interview was videotaped, transcribed, and later analyzed according to the LTs. At the conclusion of the data collection, retrospective analyses were conducted, and longitudinal accounts of growth were generated.

Results and Discussion

During the four-year teaching experiment, the research team posed 55 area tasks within 15 interviews and 48 volume tasks in 11 interviews. This report focuses on an illustrative subset of these tasks, as well as from a 33-item open-response assessment to demonstrate Drew's longitudinal growth across area and volume.

Grade 2: Initial Assessment

In the Spring of Grade 2, the research team administered an open-response assessment designed to elicit his ways of identifying and operating on units across area and volume contexts. For one of the area items, Drew was shown an image of a rectangle printed next to a square unit and asked to draw how it would look if he had a whole bunch of those squares to cover the rectangle. He drew 10 individual squares along two rows of the top of the rectangle and said, "just like that to fill it up" (Figure 1). When asked how many he thought he would need to cover the rectangle, he said "20." Drew did not need to complete his drawing; he was able to imagine repeating square units of area to cover the rectangle.

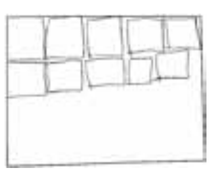

Figure 1: Drew's Drawing

Figure 2: Inch Cube and 2 x 3 x 2 in. Prism

To get a sense of Drew's attention to unit in the context of volume measurement, the interviewer gave him an inch cube and a 2 x 3 x 2 in. rectangular prism and asked how many cubes would be as big as the (2 x 3 x 2) block of wood (Figure 2). Drew, tapped the inch cube on the visible squares on all of the faces of the rectangular prism (turning it to touch all of the

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squares on all of the faces) and answered “25.” Although he had both the cube and the rectangular prism in his hands, he attended to squares on the faces rather than cubes in his count.

In Grade 2, Drew’s unit concepts in an area measurement context were more sophisticated than his unit concepts in volume measurement. His representation (drawing) for the tiling of a rectangle had some errors of alignment, his rows contained the same number of units, and he drew individual approximately the same size squares one-by-one (Figure 1). This suggests that Drew was thinking about repeating individual units of area (without considering composite units). In the context of volume measurement, however, the salient attribute of the rectangular prism was the squares on the faces. Although he correctly identified the square as the appropriate unit for area, he did not seem to identify the cube as the unit of measure for volume on the initial assessment. Drew’s nascent sense of unit observed here on the initial assessment evolved to become more sophisticated in Grade 3 in the contexts of both area and volume measurement.

Grade 3

During the Spring of Grade 3, Drew was presented with a 2 x 2 in. square (Figure 3) and asked to find the area. Drew said, “I can’t solve the area because in our math boxes in our math books, there’s like boxes so we count the boxes.” He was then given a 3 x 4 in. rectangle and asked what kind of tool he needed to be able to find the area. He said he needed “a ruler, I think.” He placed the ruler along the 4-in. side and noted it was 10 cm. He then placed the ruler along the adjacent side and used the numbered tick marks on the ruler to guide his drawing of tick marks (Figure 4). Drew explained that he used centimeters (as opposed to making use of the given number labels) because the boxes in his math books are about 1 centimeter tall and wide. He explained, “there’s a row of boxes between each one, so 10, 20, 30, 40, 50, 60, 70.” When asked to draw one of the 70, he drew a row of 10 by extending a single line across the rectangle and used the numbered tick marks on the ruler to guide his placement of segments to subdivide the row into 10 square centimeters. Drew constructed a composite unit and mentally operated on it to determine the area of the 3 x 4 in. rectangle.

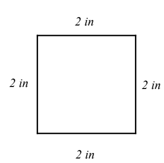

Figure 3: 2 x 2 in. square

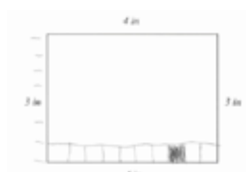

Figure 4: Drew’s drawing

Figure 5: Prisms *a*, *b*, and *c*

Also in the Spring of Grade 3, Drew was presented with a task involving unit ratio comparisons and capacity. After comparing prism *a* to *b* and *b* to *c* by pouring water (Figure 5), he said he “figured out a pattern.” Drew explained, “it would take two (prism *a*) to get into this one (prism *b*), and two of these (prism *b*) to get into that one (prism *c*).” The interviewer asked, if this (prism *a*) is one, what is this one (prism *b*)? Drew correctly said two. The interviewer then asked, if this (prism *a*) is one, what is this one (prism *c*)? He incorrectly said three. Drew explained that he thought the answer was three “because that (prism *a*) would fill up two (prism *b*); that’s (prism *c*) a unit bigger and that would fill up three.” Rather than compare the prisms based on reference to a unit, he ranked them in order from smallest to largest by capacity.

In Grade 3, Drew’s thinking about unit in area measurement was influenced by the tasks he had seen in his math book that involved counting squares that looked like square centimeters in gridded rectangles. When using the ruler to help him determine the area of the 4 x 3 in. rectangle, he attended to two dimensions and related linear centimeters to square centimeter units of area,

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but did not make use of existing number labels (Figure 5). He did not need to draw the entire array in order to operate on composite units (a row of 10) by skip counting; this is the first time we saw Drew think about repeating composite units in a systematic way.

In terms of capacity (Figure 5), his ability to identify and operate on units was limited to situations involving simple doubling. He recognized a one to two ratio for pairs of prisms (a:b is two and b:c is two), but generalized this "pattern" to an ordinal ranking when comparing three prisms (a:b is two and b:c is two, so a:c is three). That is, he did not make ratio comparisons based on reference to a unit when comparing three prisms simultaneously. Drew showed more sophisticated sense of unit in area, attending to composite units, than he did in volume in Grade 3.

Grade 4

Grade 4 was a time for important shifts in his ways of thinking about units for both area and volume. During the Fall of Grade 4 Drew was presented with an image of a 1 x 3 unit rectangle and a rectilinear figure (Figure 6) and asked, "How many grey tiles would fit inside this one (the rectilinear figure)." Drew said, "the top is nine units, and the…all the way across…not like up and down…so you can fit three units on the top…" He discussed first covering the bottom, left and right sides, and then the middle. The interviewer asked him to draw some of the units (Figure 7). Drew then explained, "I kind of just did three across the whole thing and then one, two, three, four, five," pointing to each of the five drawn units in a column. He then traced with his fingers (Figure 8) to indicate the placement of five columns of five units, using the width of his fingers to guide the placement of these columns, saying "one, two, three, four, five…five times five is 25." Once he realized the column structure, he used multiplication to curtail his process of counting by composite units (columns) to structure the rectilinear region.

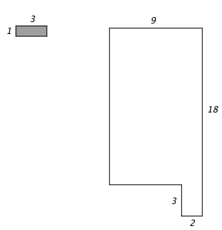


Figure 6: Rectangular Tile Task

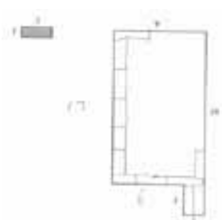


Figure 7: Drew's Drawing

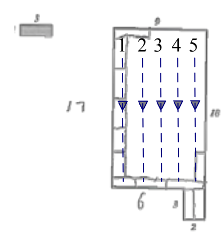
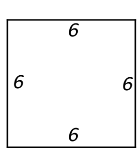


Figure 8: Drew's Finger Tracing

Two months later Drew was provided with an image of a 6 x 6 unit square and asked to determine the area (Figure 9). He traced around the square inside the boundary and explained, "I was trying to do it like painting around the outside first…so that would be like 24 centimeters already…" He went on tracing and explaining "half of 24 is 12…so 24 plus 12 is 36 I think…and then half of 12 would be 6…so 42…and then half of 6 would be 3…so that would be 45...and I think maybe it would cover it all up by then." Figure 10 illustrates Drew's border strategy, or "painting," as indicated by his finger tracing. He imagined square "ring" composites of area units with each successive becoming repeatedly smaller (by half). He was thinking about repeating composites that were the same shape, but not the same size in terms of the number of units.



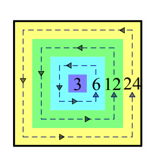


Figure 9: 6 by 6 Unit Square Task **Figure 10: Drew's Border Strategy for Area**

One week later Drew again applied his border strategy when comparing a 5 x 8 unit rectangle with a 4 x 9 unit rectangle by area (with linear dimensions indicated by tick marks). He counted around the border of each rectangle to determine the number of square units needed to cover just inside the border and divided this number in half to determine the number of units needed to cover the middle. He explained that this was just one way of finding area and that he could also multiply the length and the width. For Drew, the border strategy and multiplying length by width were two unrelated methods for determining area. Based on his discussion of this task, he did not relate the multiplication of length by width to a structuring of the rectangles by composite units.

During the early Spring of Grade 4 Drew was presented with the images of a net and an open top box (Figure 11). The interviewer explained that the net could be made by cutting along the edges of the box and flattening it and asked, "How many cubes would it take to fill this box? Drew pointed and counted the cubes on the bottom of the net one-by-one in columns in an 'S' pattern (Figure 11). He explained that he needed three "stacks" of 24 or 24 times 3 to fill the box. That is, he operated on a composite unit, a layer, using multiplication to determine the number of cubes that he would need to fill the box.

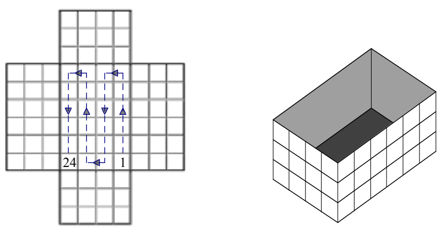


Figure 11: Net and Box

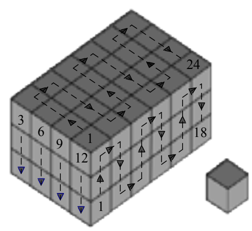


Figure 12: Border Strategy for Volume

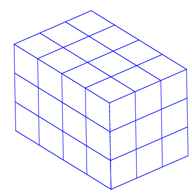


Figure 13: 4 x 3 x 3 Prism

During the same interview, he was presented with an image of a 4 x 6 x 3 unit rectangular prism printed next to a unit cube (Figure 12) and asked, "How many of these blocks (pointing to the unit cube) would it take to make that (pointing to the 4 x 6 x 3)?" Drew counted all of the visible cubes on the front "wall" one-by-one in columns to get 18 cubes. He explained that there were another 18 blocks on the opposite side that he could not see. He then skip counted by threes, saying "three, six, nine, twelve" to determine the number of cubes on the adjacent "wall," and explained that there were 12 more cubes in another "wall" that he could not see. He multiplied 18 by 2 to get 36 and 12 by 2 to get 24 and added 36 + 24 = 60. He then counted the visible squares on the top of the figure one-by-one in rows to get 24. He added the 24 to 60 to find 84. When the interviewer asked him how he got his answer, he explained that he had counted the whole thing, but then decided that he had double counted some of the blocks. He said he thought that he should not have counted the top, so he decided that the first part of his answer, which was 60 was correct; however, he said he could not remember how he got 60. Drew's ways of operating on units in the context of volume changed according to task representation. Without the perceptual support of the net and an open-top box to imagine filling up, Drew attended to individual units on the faces of the rectangular prism rather than composite units (layers).

Five months later, Drew was presented with an image of a gridded 4 x 3 x 3 unit rectangular prism (Figure 13) and the interviewer asked, "How many small cubes like the one shown are in this large block?" Drew said, "I'm going to try a different way," which he clarified to mean that

he was not going to count each unit individually. He counted 12 visible squares on the 4 x 3 face and 9 visible squares on the adjacent 3 x 3 face and multiplied 12 by 9, and when he was asked how he knew to multiply 12 by 9 he said, “I heard of that method kind of…it’s usually with tiles, though.” The interviewer replied that she had heard of that method for area and asked if he thought the same thing would work here. Drew said, “I think so.” When asked if he thought 108 fit there, he said “um…no…it doesn’t look like it” and said he thought it looked like it should be less. Drew’s application of multiplication here was not connected to a structuring of the rectangular prism by iterating composites (layers).

During Grade 4 Drew exhibited variability in his ways of thinking about units in both area and volume measurement contexts. He operated on composite units using multiplication for some area and volume measurement tasks. This notion of multiplication is repeated addition. Although he operated on a column as a composite unit using multiplication in an area measurement task (Figure 8), he did not consistently relate linear and area units (Kara, Eames, Miller, Cullen, & Barrett, 2011). That is, he did not consistently apply the concept that the length of a side determines the number of area units that fit along the side. As his sense of a composite evolved throughout Grade 4, he generalized his operations on units in an alternative way. For example, he invented the border strategy (Figure 10) as an alternative to multiplying length by width when determining the area of a rectangle. His willingness to suggest that multiplying length by width would give the same answer as his border strategy suggests that he did not seem to see a conceptual link between his use of multiplication to model the mental iteration of a composite unit (as depicted in Figure 8) and the multiplication of length by width.

Drew extended his alterative way of operating on units in area measurement contexts into contexts involving volume during Grade 4. For example, his attention to surface area (Figure 12) seemed to be an extension of his border strategy (Figure 10) from area to volume. At the end of Grade 4, he adapted his multiplication of length by width strategy for area to make sense for him in the context of volume measurement. Instead of multiplying lengths of perpendicular adjacent sides to produce a measure of area, he multiplied surface areas of perpendicular adjacent faces to produce a measure of volume. The variability in Drew’s strategy use in Grade 4 (operating on composite, counting around the border, and using multiplication) suggests that his unit concepts were still under development in both area and volume measurement contexts at this time.

Grade 5

Ten months later, during the Fall of Grade 5, Drew was given a 5 x 7 in. rectangle and asked to compare the area of the tile to the area of the larger rectangle. He noted, “5 of these would take up the side rows (pointing to the left and rightmost columns)…and…um…7 of these would take up the longer (pointing to a bottom row of 7).” He then said that the larger rectangle was 35 times bigger than the unit. When asked how he knew that multiplication would tell him how many units would fill the rectangle, he said, “I don’t know…I just kinda learned it.” The interviewer asked him to draw units, and he drew a column of 5 on the right-hand-side, using the unit tick marks on the ruler to guide his placement of units. Before completing his drawing, he expressed doubt that his answer of 35 obtained by multiplying was correct because he noticed that the corner tiles “took up two spaces.” To check, he finished drawing around the border and then covered the interior by drawing the squares by columns. After he finished his drawing he began counting the units one-by-one in the same pattern he described in his tracing in Grade 4 (Figure 15). While counting, Drew suddenly noted that there were five tiles in a column and switched to counting by 5s saying, “5, 10, 15, 20, 25, 30, 35.” He realized that counting by columns produced the same answer as multiplying five by seven.

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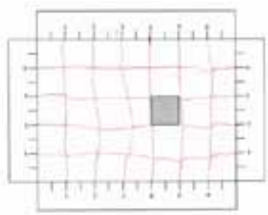

Figure 14: 5 x 7 in. Rectangle

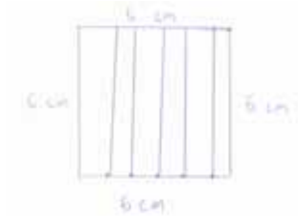

Figure 15: Drew's Rectangle

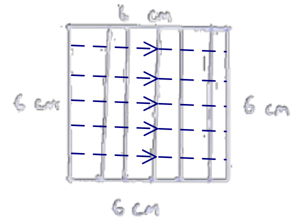

Figure 16: Drew's Gestures

During the Spring of Grade 5, Drew was asked to use a ruler to draw a rectangle with an area of 36 square centimeters. Drew explained that he would have to think backwards from the area formula, and used the ruler to draw the border of a 6 x 6 cm square. After creating his drawing, he said that he thought his square might have too many units. He checked by lining up the ruler with one side and using the numbered tick marks to guide the placement of parallel column line segments (Figure 15). The interviewer interrupted him and asked if he still thought that his square had too many units. Drew gestured with his hand where the parallel row segments would intersect the column segments to form a rectangular array structure of identical square units (Figure 16). He explained that the square has six columns of six square centimeters and six rows of six square centimeters. In the context of this task Drew connected the multiplication of length by width to the structuring of the square with composite units (rows and columns).

Two weeks later, Drew was presented with a wrapped rectangular prism and an inch cube (Figure 17) and he was asked to compare the volume of the inch cube to the volume of prism.

Figure 17: Wrapped Rectangular Prism and Inch Cube

He iterated the inch cube along the length, width, and height of the prism and found that the dimensions were 4 x 3 x 3 in. He said, "I think to find the volume what you're gonna want to do is the width times the height times the length." He noted that 3 times 4 is 12 and 12 times 3 is 36. When asked to draw where 12 would go, he drew a row of 4 on the 4 x 3 in. side and explained that the side would contain three rows of four. For the adjacent 3 x 3 in. side, he drew one unit in the corner and said, "three times three so it would take 9 blocks to fill this side." For the total, he said he would need to do 12 times 9, which the interviewer told him was 108. He then said, "it wouldn't take up 108…I think it will take up 36." To explain his answer, he said "I did 4 times 3 times 3…36…so that's the width times the height times the length." The interviewer drew his attention back to the 4 in x 3 in side by asking how many 12s there were. Drew pointed to the 4 x 3 in. side and explained, "[it would take] 12 blocks to fill that…and then 3 to fill that side…it was actually 9…but the width would be 3 so it would be 3 times 4 times 3...it just goes back to the length times width times height." Although Drew trusted his answer from multiplying length by width by height, he did not clearly connect his application of multiplication by the structuring of the rectangular prism by composite units (layers).

Drew abandoned his border strategy early in Grade 5 because realized that multiplying length by width was the same as operating on a composite unit by skip counting for area (Figure 14). He consistently related linear and area units and used this relationship to develop a drawing strategy for checking his thinking when multiplying length by width. He used the numbered tick marks on a ruler to guide his drawing of parallel row and column line segments to produce a grid so that he could check to see if multiplying length by width produced the correct answer (Figures 15 and 16). He also realized that his adaptation of multiplying length by width for volume (as the multiplication of areas of perpendicular adjacent faces) did not produce the same answer as the multiplication of length by width by height. During Grade 5, Drew showed evidence that he learned that multiplication *does* produce a measure of area or volume; however, he may not yet have understood *why* multiplication produces a measure of area or volume.

Conclusions and Implications

Drew gained sophistication for measuring space more slowly than the other six students in this cohort, perhaps because he sought to systematize his ways of operating on composite units in alternative ways. However, the results of this longitudinal work, which are consistent with prior cross-sectional research (Battista & Clements, 1996; Battista, Clements, Arnoff, Battista, & Borrow, 1998; Outhred & Mitchelmore, 2000), indicate that Drew's understanding of the conceptual link between the iteration of a composite unit and the application of multiplication in both area and volume measurement did increase from Grade 3 to Grade 5. We believe it is important to note that Drew exhibited growth in terms of his use of unit in an area measurement context; and this often precipitated growth in volume measurement. In the current Common Core State Standards for Mathematics, measuring areas of rectangles is introduced and mastered in Grade 3 and measuring volume of rectangular prisms is introduced and mastered in Grade 5. This study adds to the body of research that suggests introducing and mastering these concepts in a single grade level may not be developmentally appropriate for children. Therefore, we recommend providing early experiences for children to think about building and operating on composite units before and during instruction of formulas in a broad time span. Furthermore, Drew's thinking in terms of unit in both area and volume varied in different task representations and use of tools. We suggest that during the instruction of area and volume measurement, tasks should vary in terms of representation.

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